ARL 63-83

A ONE-DIMENSIONAL MODEL FOR COMPRESSIBLE FLOW IN THE TRAVELING WAVE PUMP

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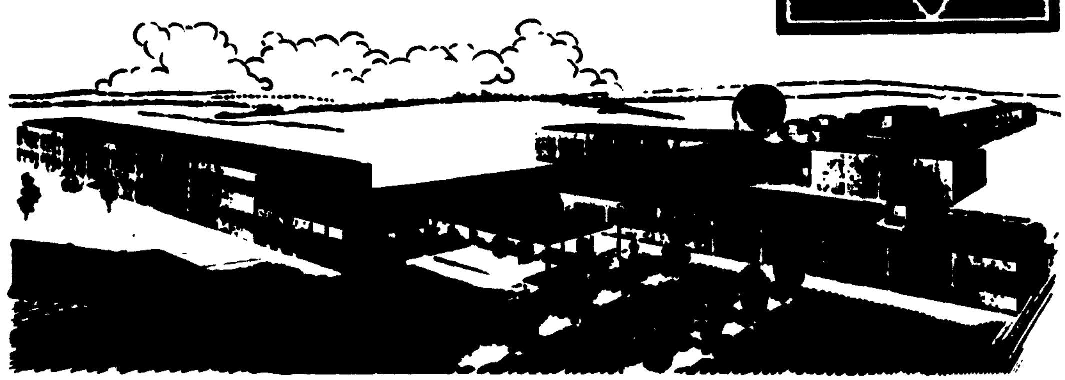
AERONAUTICAL RESEARCH LABORATORIES
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE



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300 - September 1963 - 162 - 3-25

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Contract AF 33(657)-7975
Project 7065
Task 706501

AERONAUTICAL RESEARCH LABORATORIES
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UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This interim technical report was prepared by the Aerophysics Laboratory of the Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts on Contract AF 33(657)-7975 for the Aeronautical Research Laboratories, Office of Aerospace Research, United States Air Force. The research reported herein was accomplished on Task 7065-01, "Fluid Dynamics Facilities Research" of Project 7065, "Aerospace Simulation Techniques Research" under the technical cognizance of Mr. Robert G. Dunn of the Fluid Dynamics Facilities Laboratory of ARL.

ABSTRACT

One possible technique for accelerating a plasma to a high Mach number is that of passing a magnetic field through the plasma. This technique, called the traveling wave pump, is characterized by a complicated set of differential equations. These equations have been approximated by a one-dimensional steady state form. Several approximate integrals are found for the one-dimensional equations. The results indicate that the viscosity losses in the system are not excessive. The results also indicate that the inlet velocity profile should be as uniform as possible and that the magnitude of the inlet velocity should exceed $\gamma-1/\gamma$ times the wave speed by an appreciable amount.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

A	Area of TWP duct
a	$= -1 + \phi\delta$
a_1	Characteristic length defined in text
$\vec{B}$	Magnetic field intensity
B_e	Equivalent average field strength
C_1, C_2	Integration constants defined in text
C_f	Friction coefficient $= \dfrac{\text{wall drag per unit area}}{\frac{1}{2}\rho u^2}$
C_p	Specific heat at constant pressure
c	Phase velocity of pumping coil $= \dfrac{\omega}{k}$
D	Mean hydraulic diameter $= \dfrac{4\,(\text{area})}{\text{wetted perimeter}}$
$\vec{E}$	Electric field strength
$\vec{F}$	Force per unit volume
f	Friction factor $= \dfrac{\text{wall shear stress}}{\frac{1}{2}\rho u^2}$
I	Current in coil
I_N	Bessel function of the first kind of imaginary argument of order N
J	Current density
K_N	Bessel function of the second kind of imaginary argument of order N
k	Propagation constant of TWP coil
M	Local Mach number
N	Number of turns of TWP coil per unit length
P	Pressure or equivalent nondimensional pressure
P_T	Total power additional per unit volume

dQ	Total energy added to control volume per unit mass of fluid in control volume including work and heat
r, θ, z	Circular cylindrical coordinates
R	Gas constant
Re	Fluid Reynolds number
R_o	Tube radius or coil inlet radius
t	Time
T	Period of a periodic function, temperature
u	Mass average velocity, (equivalent nondimensional velocity)
V	Local velocity
x	Expansion variable equivalent to 0^{th} order velocity solution
X	Total body force acting on control volume in direction opposed to u
z	Axial distance
α	Velocity weighting function defined in text
γ	Ratio of specific heats
δ	$= \dfrac{\frac{R}{c_p}}{1 - \frac{\alpha R}{c_p}}$
ϵ	Logarithmic expansion parameter defined in Appendix II
η	Viscosity
λ	Wavelength of traveling wave on pumping coil
μ	Magnetic permeability
ρ	Mass density
σ	Electrical conductivity or molecular diameter
ϕ	Nondimensional friction defined in text

LIST OF SYMBOLS (concluded)

ψ	Nondimensional area change defined in text
Ω	$= \kappa(z - ct)$
ω	Angular frequency
$(\ )^*$	Nondimensional variable (used only when nondimensional variables are first introduced)
$(\ )_{in}$	Condition at inlet to TWP duct

CHAPTER I

ANALYSIS

The complexity of the flow in the TWP accelerator seems to require a simpler procedure for a working approximation. Since the TWP is an A.C. apparatus the first step is that of reducing the equations to the form of steady flow. The one-fluid model will be used. The next step is that of reducing the equations to one-dimensional flow. These steps which give simpler, though more approximate set of equations are outlined below.

Consider first the continuity of mass,

$$\frac{\partial \rho}{\partial t} + div\,(\rho \vec{v}) = 0 \qquad (1)$$

where

ρ = mass density

$\vec{v}$ = mass velocity

t = time.

If this equation is integrated over a volume (this volume is not time dependent) and the divergence is converted to a surface integral, one obtains

$$\int_{vol} \frac{\partial \rho}{\partial t}\, dV + \int_{S} \rho \vec{v} \cdot d\vec{s} = 0 \qquad (2)$$

Manuscript released by the authors December 1962 for publication as an ARL Technical Documentary Report.

Equation (2) is then integrated over a period, i.e.,

$$\int_t^{t+T} dt \left(\int_{vol} \frac{\partial \rho}{\partial t} dV \right) + \int_t^{t+T} dt \int_S \rho \bar{V} \cdot d\bar{S} = 0 \qquad (3)$$

and if

$$\int_{vol} \frac{\partial \rho}{\partial t} dV \Big|_t = \int_{vol} \frac{\partial \rho}{\partial t} dV \Big|_{t+T}$$

then this term vanishes. The second term is evaluated over a stream tube of cross section $A(z)$ lying between z and $z + dz$. Thus the continuity reduces to

$$\rho u A = \text{Const.} \quad \text{or} \qquad (4a)$$

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \qquad (4b)$$

This volume corresponds to the control volume used by Shapiro.

Similar treatment of the momentum equation results in the average one dimensional equation,

$$\frac{dP}{P} + \frac{dX}{PA} + \frac{\gamma M^2}{2} \left\{ 4f \frac{dz}{D} + \frac{du^2}{u^2} \right\} = 0$$

where

M = local Mach number

$M = u / \sqrt{\gamma R T}$

P = pressure

γ = ratio of specific heats

$d\underline{X}$ = total body force acting on the control volume in direction opposed to u

$$d\underline{\bar{X}} = \left(\frac{1}{\lambda}\int_{z}^{z+\lambda}\left[\frac{1}{T}\int_{t}^{t+T} dt\left(\int_{A}\vec{J}\times\vec{B}\right)_{z} dA\right]dz\right)dz$$

D = hydraulic diameter

$D = 4\,\dfrac{\text{area of duct}}{\text{perimeter}}$

f = friction factor

$f = \tau_w/q\ ;\ q = \frac{1}{2}\rho u^2$

The energy equation is also averaged over the period and gives

$$\frac{dQ}{C_p T} = \left[\frac{dT}{T} + \frac{\gamma-1}{2} M^2 \frac{du^2}{u^2}\right] \tag{6}$$

where

C_p = specific heat at constant pressure

dQ = work and heat added to the contents of the control volume per unit mass of contents

The state equation is

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dR}{R} + \frac{dT}{T} \tag{7}$$

The next step is that of evaluating the averaging integrals, which can be calculated from the results given in Ref. 2 after computing the electric and magnetic fields. When the magnetic Reynolds number per unit length based on the slip velocity ($\sigma\mu(c-u)$), is small, the attenuation of the coil is small and $\frac{\sigma\mu c}{\omega}(c-u)$ is small. The arguments of the Bessel functions describing the fields below are real, with $K = \frac{\omega}{C}$. The fields are given below for this case:

$$B_r = \mu N I \; I_1(Kr) \; G(KR_0) \; \sin \Omega$$

$$B_z = \mu N I \; I_0(Kr) \; G(KR_0) \; \cos \Omega$$

$$E_\theta = -C \mu N I \; G(KR_0) \; I_1(Kr) \; \sin \Omega$$

$$J_\theta = -\sigma \mu N I \; I_1(Kr) \, G(KR_0) \, (C - V_z)$$

The force per unit volume is

$$\vec{F} = \vec{J} \times \vec{B} = \hat{a}_r J_\theta B_z - \hat{a}_z B_r J_\theta$$

$$F_r = -\sigma (\mu N I)^2 G^2(KR_0) \, I_0(Kr) \, I_1(Kr) \sin \Omega \cos \Omega \, (C - V_z)$$

$$F_z = \sigma (\mu N I)^2 \, I_1{}^2(Kr) \, G^2(KR_0) \sin^2 \Omega \, (C - V_z)$$

The total electrical energy added per unit volume, per unit time (i.e., the total power addition) is

$$P_T = \vec{E} \cdot \vec{J} = \sigma (\mu N I)^2 \, I_1{}^2(Kr) G^2(KR_0) \sin^2 \Omega \; C(C - V_z)$$

where

$$K = \omega / C \; , \quad \Omega = K(z - Ct)$$

and

$$G(KR_0) = \frac{K_1(KR_0)}{K_1(KR_0)\,I_0(KR_0) - K_0(KR_0)\,I_1(KR_0)}$$

where $I_{(N)}$ and $K_{(N)}$ are Bessel functions of the first and second kind, respectively, of order N having imaginary argument.

For $KR \longrightarrow 0$ $\qquad G(KR) \longrightarrow \dfrac{2}{2 + (KR)^2 \ln(KR)} \approx 1$

The force on the control volume of length z then becomes

$$\bar{X} = -\frac{z}{\lambda}\int_z^{z+\lambda} dz \, \frac{1}{T}\int_t^{t+T} 2\pi \int_0^{R_0} G^2(KR_0)\,\sigma(\mu N I)^2 [I_1^2(Kr)(C - V_z) \sin^2 K(z-ct)\, r\,dr$$

$$= -\sigma(\mu N I)^2 \frac{z}{4} \cdot 2\pi \int_0^{R_0} G^2(KR_0)\, I_1^2(Kr)(C - V_z(r))\, r\,dr$$

$$= \frac{1}{4}\,\sigma(\mu N I)^2\,[\pi R_0^2][I_1^2(KR) - I_0(KR)\,I_2(KR)] \text{ times}$$

$$G^2(KR_0) \cdot [C - \alpha u]\, z$$

$$\text{where } \alpha u = \frac{2\pi \int_0^{R_0} I_1^2(Kr)\, V_z(r)\, r\,dr}{\pi R_0^2 [I_1^2(KR) - I_0(KR)\, I_2(KR)]}$$

α is a weighting function which adjusts u, the mass average velocity to account for the variation in magnetic field across the tube. In the case of slug velocity $\alpha = 1$.

Let

$$F(KR) = [I_1^2(KR) - I_0(KR)\, I_2(KR)]$$

$$\text{Then as } KR \rightarrow \infty, \quad F(KR) \rightarrow \frac{5e^{2KR}}{8\pi (KR)^2}$$

and

$$d\mathcal{I} = \frac{-\sigma(\mu N I)^2}{4} G^2(KR_0)\, \pi R_0^2 F(KR_0)(c - \alpha u)\, dz$$

and

$$dQ = \frac{1}{\lambda}\int_0^{\lambda} dz \left[\frac{1}{T}\int_0^{T} dt \left(\int_0^{R_0} 2\pi r\, \frac{\vec{E}\cdot\vec{J}}{\rho u A}\, dr\right)\right] dz$$

$$= \frac{\sigma(\mu N I)^2}{4\rho u} G^2(KR_0)\, F(KR_0)\, c(c - \alpha u)\; dz$$

A one dimensional form has also been deduced by Williams in Ref. 3. However he fails to account for α .

Equations (4), (5), (6), and (7) describe the process to be analyzed. Since these equations constitute a non linear system, it seems most practical to follow Dahlberg's suggestion in Ref. 4 and study the phase plane in search of closed form solutions.

The momentum equation then becomes:

$$\frac{dP}{dz} + \frac{1}{A}\frac{d\mathcal{I}}{dz} + \gamma M^2 P \left\{2\frac{f}{D} + \frac{1}{u}\frac{du}{dz}\right\} = 0 \qquad (8)$$

Using $\rho u^2 = \gamma P M^2$

$$\rho u \frac{du}{dz} + \frac{dP}{dz} = \frac{\sigma(\mu N I)^2}{4} F(\kappa R_o) G^2(\kappa R_o)(C - \alpha u) - 2\rho u^2 \frac{f}{D}$$

or

$$\frac{dP}{dz} = \frac{\sigma(\mu N I)^2}{4} F(\kappa R_o) G^2(\kappa R_o)(C - \alpha u) - 2\rho u^2 \frac{f}{D} - \rho u \frac{du}{dz} \qquad (9)$$

The energy equation (6), neglecting the ionization term, using the equation of state and expanding $\frac{du^2}{u^2} = \frac{2du}{u}$, becomes

$$dQ = C_p \frac{dT}{T} \frac{P}{\rho R} + \frac{C_p}{R} \frac{\gamma - 1}{\gamma} u\,du$$

From (4b) and (7)

$$\frac{dT}{T} = \frac{dP}{P} - \frac{d\rho}{\rho} = \frac{dP}{P} + \frac{du}{u} + \frac{dA}{A}$$

so

$$\frac{dQ}{dz} \rho u = \frac{C_p}{R} \left[u \frac{dP}{dz} + P \frac{du}{dz} + \frac{Pu}{A} \frac{dA}{dz} + \frac{\gamma - 1}{\gamma} \rho u^2 \frac{du}{dz} \right]$$

leading to

$$\left(\rho u^2 + \frac{C_p}{R}P\right)\frac{du}{dz} + u\frac{C_p}{R}\frac{dP}{dz} = -\frac{C_p}{R}\frac{Pu}{A}\frac{dA}{dz}$$

$$+\frac{\sigma(\mu NI)^2}{4}F(KR_0)G^2(KR_0)C(C-\alpha u) \qquad (10)$$

Solving for $\dfrac{du}{dz}$

$$\frac{du}{dz} = \frac{\left\{\dfrac{uC_p}{R}\left[\dfrac{P}{A}\dfrac{dA}{dz} - 2\rho u^2\dfrac{f}{D} + \dfrac{\sigma(\mu NI)^2}{4}F(KR_0)G^2(KR_0)(C-\alpha u)\right] - \dfrac{\sigma(\mu NI)^2}{4}F(KR_0)G^2(KR_0)C(C-\alpha u)\right\}}{\left(\dfrac{C_p}{R}-1\right)\rho u^2 - \dfrac{C_p}{2}P} \qquad (11)$$

Substituting in (9) and collecting terms

$$\frac{dP}{dz} = \frac{\left\{2\rho u^2\dfrac{f}{D}\left[\rho u^2 + \dfrac{C_p}{R}P\right] - \rho u^2\dfrac{C_p}{R}\dfrac{P}{A}\dfrac{dA}{dz} - \left[\rho u^2 + \dfrac{C_p}{R}P\right]\dfrac{\sigma(\mu NI)^2}{4}F(KR_0)G^2(KR_0)(C-\alpha u) + \rho u\sigma\dfrac{(\mu NI)^2}{4}F(KR_0)G^2(KR_0)C(C-\alpha u)\right\}}{\left[\dfrac{C_p}{R}-1\right]\rho u^2 - \left(\dfrac{C_p}{R}\right)P} \qquad (12)$$

Examination of these two equations indicates that

a. there is a critical line corresponding to

$$\rho u^2\left(\frac{c_p}{R}-1\right)-\frac{c_p}{R}\,p=0$$

or since ρ, and P are not generally zero this line corresponds to $M=1$

b. there are singular points where $\frac{du}{dx}=0$. This occurs if

i. $u=\frac{c}{\alpha}$ and

ii. $$\frac{P}{A}\frac{dA}{dx}=2\rho u^2\frac{f}{D}$$

When this singular point is located within the region of interest it offers a possible termination for the process.

CHAPTER II

DISCUSSION

First consider the equations (11) and (12) in the physical plane. These two equations can be integrated numerically from given conditions at the inlet, to the outlet of a TWP device once its geometry has been specified.

In order to find the circumstances which will lead to an increase in Mach number, consider the following expression from Ref. 1 which follows from the momentum state continuity and energy equations.

$$2\frac{dM}{M} = -2\left(1+\frac{\gamma-1}{2}M^2\right)\frac{dA}{A} + \frac{1+\gamma M^2}{1-M^2}\,\frac{dQ}{C_p T}$$

$$+ \frac{\gamma M^2\left(1+\frac{\gamma-1}{2}M^2\right)}{1-M^2}\left(4f\frac{dz}{D} + \frac{dX}{\frac{1}{2}\gamma P A M^2}\right)$$

This can be rewritten as:

$$\frac{1}{M}\frac{dM}{dz} = \frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\left(-\frac{dA}{dz}\frac{1}{A} + 2\gamma M^2\frac{f}{D}\right)$$

$$+ \frac{1+\gamma M^2}{1-M^2}\,\frac{dQ}{dz}\,\frac{1}{2C_p T} + \frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\,\frac{dX}{dz}\,\frac{1}{PA} \qquad (13)$$

Consider the case when the friction cancels the area change. This case exhibits the effects of the magnetic power addition only. For $\frac{dM}{dz}$ to be positive

$$\left(1+\frac{\gamma-1}{2}M^2\right) > (1+\gamma M^2)\frac{1}{2}\frac{c}{\alpha u}\frac{\gamma-1}{\gamma} \qquad (13a)$$

For this case $\frac{dM}{dz} = 0$ lines for constant values of γ are shown in Fig. 1. These lines separate the plane into $\frac{dM}{dz} < 0$ and $\frac{dM}{dz} > 0$ regions. Also for this case if $\frac{du}{dz}$ is also to be positive, u/C must be greater than $\frac{\gamma-1}{\gamma}$.

This constraint is considerably less stringent than the previous one,($\frac{du}{dz}$ can be positive when $\frac{dM}{dz}$ is 0 or negative) hence it will not be plotted.

Dahlberg (Ref. 4) has shown that Eqs. (11) and (12) can be put into a symmetrical form by the proper choice of nondimensional variables.

Choose $u^* = \frac{u}{C}$ and $p^* = \frac{\frac{c_p}{R}p}{\rho u C}$ as new variables and (14) express the equations in terms of two nondimensional parameters:

$$\nu = \frac{1}{A}\frac{dA}{dz}\frac{\rho u}{B_e^2\sigma\frac{c_p}{R}} \qquad \text{and} \qquad \phi = \frac{2\rho u}{B_e^2\sigma}\frac{f}{D} \qquad (14a)$$

and the characteristic length $a_1 = \frac{\rho u}{B_e^2\sigma\frac{c_p}{R}}$

For the case of constant phase velocity C and gas properties the equations become

$$a_1\frac{du^*}{dz} = \frac{(\alpha u^*-1)\left(u^*-\frac{R}{c_p}\right)+u^*(\phi u^*-\nu p^*)}{p^*-\left(\frac{c_p}{R}-1\right)u^*} \qquad (15)$$

$$a_1 \frac{dP^*}{dz} = \frac{(\alpha u^* - 1)(P^* + u^* - 1) + (P^* + u^*)(\phi u^* - \psi P^*)}{P^* - \left(-1 + \frac{c_p}{R}\right) u^*} \qquad (16)$$

For the case with cylindrical symmetry B_e is the equivalent average field.

$$B_e^{\,2} = \frac{(\mu N I)^2}{4} F(KR_0)\, G^2(KR_0)$$

In terms of these variables at the singular point mentioned above, $\frac{dP^*}{dz} = \frac{du^*}{dz} = 0$, i.e., the nondimensional pressure and velocity become constant giving in a sense a "fully developed," stationary solution, although, of course, the dimensional pressure must decrease along the duct to balance the friction under this condition. The nondimensional variables thus give a much better description of the termination point of the process.

This singular point toward which the magnetic pumping drives the flow, must be located in the supersonic part of the phase plane and the friction must not be too high if supersonic acceleration is to occur. These limits are expressed by regions i and k of the $\phi - \psi$ plot of Ref. 4 and impose the constraints:

a. $\frac{1}{2\gamma A} \frac{dA}{dz} \geq \frac{f}{D}$ and

b. $\frac{B_e^{\,2} \sigma}{8 \rho u \gamma (\gamma - 1)} \geq \frac{f}{D}$

CHAPTER III

SOLUTION IN THE PHASE PLANE

Equations (15) and (16) are identical to those of Dahlberg (Ref. 4) when the variation of the field with radius is accounted for through B_e , and α the velocity averaging factor. We follow his procedure dividing (16) by (15) to obtain

$$\frac{dP}{du} = -\frac{(\alpha u-1)(P+u-1)+(P+u)(\phi u-\psi P)}{(\alpha u-1)(u-\frac{R}{C_p})+u(\phi u-\psi P)} \qquad (17)$$

The *'s have been omitted; however the nondimensional variables (defined in (13) and (14)) are still being used.

Following Lighthill (Ref. 5) let

$$P = P_0(x) + \psi P_1(x) + \ldots$$

and (18)

$$u = x + \psi u_1(x) + \ldots$$

$$\frac{dP}{du} = \frac{\frac{dP}{dx}}{\frac{du}{dx}}$$

$$\frac{du}{dx} = 1 + \psi u_1' \ldots$$

(19)

$$\frac{dP}{dx} = P_0' + \psi P_1' \ldots$$

(17) can then be re-written

$$\frac{dP}{dx}\left[(\alpha u-1)\left(u-\frac{R}{C_p}\right)+u\,(\phi u-\psi P)\right]=(-)\,\frac{du}{dx}\left[(\alpha u-1)(P+u-1)+\right.$$

$$\left.+(P+u)(\phi u-\psi P)\right] \qquad (20)$$

Substituting (18) and (19) into (20) and equating powers of Ψ, the nondimensional area change parameter, we obtain the 0^{th} order equation for P_0';

$$-P_0'=\frac{(\alpha x-1)(P_0+x-1)+\phi x\,(P_0+x)}{(\alpha x-1)\left(x-\frac{R}{C_p}\right)+\phi x^2} \qquad (21)$$

and the two first order equations:

$$P_1'=-P_1\,\frac{(\phi+\alpha)\,x-1}{(\alpha x-1)\left(x-\frac{R}{C_p}\right)+\phi x^2} \qquad (22)$$

and

$$u_1'+f(x)\,u_1=g(x) \qquad (23)$$

where

$$\left.\begin{aligned} f(x)&=\frac{\alpha(P_0+x-1)+(\alpha+2\phi)x-1+\phi P_0}{(\alpha x-1)(P_0+x-1)+\phi z(P_0+z)}\\ &\quad-\frac{(\alpha+\phi)x+(\alpha-1)x-\left(1+\frac{\alpha R}{C_p}\right)}{(\alpha x-1)\left(x-\frac{R}{C_p}\right)+\phi x^2}\\ \text{and}\quad g(x)&=\frac{P_0(P_0+x)}{(\alpha x-1)(P_0+x-1)+\phi x(P_0+x)}\\ &\quad-\frac{P_0x}{(\alpha x-1)\left(x-\frac{R}{C_p}\right)+\phi x^2}\end{aligned}\right\} \qquad (23a)$$

Equation (21) is a first order, linear differential equation for P_0 which can be reduced to quadratures and integrated in the case when ϕ is small. $P_1 = 0$ is a regular solution of (22) which is chosen to satisfy the initial conditions that at $u = u_{inlet}$, $P = P_0 + \psi P_1 = P_{inlet}$, $P_1 = 0$.

After solution of (21), (23) can be reduced to quadratures, but it involves a great many terms and an approximation to the logarithmic function even for small ϕ and ψ.

a. <u>The Solution for P_0</u>

Collecting the terms in (23) we obtain

$$P_0' + \frac{(\alpha+\phi)x-1}{(\alpha+\phi)x^2 - x\left(1+\frac{\alpha R}{C_p}\right)+\frac{R}{C_p}} = -\frac{(\alpha+\phi)x^2 - x(\alpha+1)+1}{(\alpha+\phi)x^2 - x\left(1+\frac{\alpha R}{C_p}\right)+\frac{R}{C_p}} \qquad (24)$$

The solution of this equation is

$$P_0(x) = \exp(-I(x))\left[C_1 - \int \exp I(x)\,\frac{(\alpha+\phi)x^2 - x(\alpha+1)+1}{(\alpha+\phi)x^2 - x\left(1+\frac{\alpha R}{C_p}\right)+\frac{R}{C_p}}\,dx\right] \qquad (25)$$

where

$$I(x) = \int \frac{(\alpha+\phi)x-1}{(\alpha+\phi)x^2 - x\left(1+\frac{\alpha R}{C_p}\right)+\frac{R}{C_p}}\,dx$$

This can be integrated* to give

$$I(x) = \frac{1}{2} \ln \left[(\alpha+\phi)x^2 - x\left(1+\frac{\alpha R}{C_p}\right) + \frac{R}{C_p} \right]$$

$$+ \frac{1}{2\sqrt{1-\dfrac{\frac{4R}{C_p}\phi}{\left(1-\frac{\alpha R}{C_p}\right)^2}}} \ln \left[\frac{2(\alpha+\phi)x - \left(1+\frac{\alpha R}{C_p}\right) - \left(1-\frac{\alpha R}{C_p}\right)\sqrt{1-\dfrac{\frac{4R\phi}{C_p}}{\left(1-\frac{\alpha R}{C_p}\right)^2}}}{2(\alpha+\phi)x - \left(1+\frac{\alpha R}{C_p}\right) + \left(1-\frac{\alpha R}{C_p}\right)\sqrt{1-\dfrac{\frac{4R\phi}{C_p}}{\left(1-\frac{\alpha R}{C_p}\right)^2}}} \right]$$

using this result (25) becomes

$$P_0 = (A+B)^{-m}(A-B)^{-n}\left[C_1\sqrt{\alpha+\phi} - 2\int (A+B)^{m-1}(A-B)^{n-1}\left[(\alpha+\phi)x^2-(\alpha+1)x+1\right] dA\right] \tag{26}$$

where the following abbreviations have been used.

$$m = \frac{1}{2}\left[1+\frac{1}{\sqrt{1-\dfrac{\frac{4R\phi}{C_p}}{\left(1-\frac{\alpha R}{C_p}\right)^2}}}\right]$$

---

* For the Logarithmic function to be the solution, $\left(1+\frac{\alpha R}{C_p}\right)^2 > \frac{4R}{C_p}(\alpha+\phi)$ is required. Rewriting this condition as

$$\left(1-\frac{\alpha R}{C_p}\right)^2 - 4\phi\frac{R}{C_p} > 0$$

it can be seen that with

$$0 \le \alpha \le 1, \quad \frac{R}{C_p} = \frac{\gamma-1}{\gamma} \quad \text{and} \quad 1 \le \gamma \le 1.67$$

this condition is nearly always met. For example, the limit is $\phi < .225$. For the worst condition of $\alpha = 1$, $\gamma = 1.67$.

$$n = \frac{1}{2}\left[1 - \frac{1}{\sqrt{1 - \dfrac{\frac{4R\phi}{C_p}}{\left(1 - \frac{\alpha R}{C_p}\right)^2}}}\right]$$

$$A = 2(\alpha+\phi)X - \left(1 + \frac{\alpha R}{C_p}\right)$$

$$B = \left(1 - \frac{\alpha R}{C_p}\right)\sqrt{1 - \frac{4R\phi}{C_p} \Big/ \left(1 - \frac{\alpha R}{C_p}\right)^2}$$

$$A - B = 2\left[(\alpha+\phi)X - 1 + \frac{\frac{R}{C_p}\,\phi}{\left(1 - \frac{\alpha R}{C_p}\right)}\right]$$

$$A + B = 2\left[(\alpha+\phi)X - \frac{\alpha R}{C_p} - \frac{\frac{R}{C_p}\,\phi}{\left(1 - \frac{\alpha R}{C_p}\right)}\right]$$

The second term of (26) cannot be integrated in general; but when ϕ is small (see Appendix II), the expression can be integrated directly since $m \rightarrow 1$, $n \rightarrow 0$.

For $\phi \ll 1$ and absorbing a numerical factor in the constant C_1.

$$P_0 = \frac{C_1}{(\alpha+\phi)X - \frac{\alpha R}{C_p} - \frac{\phi\left(\frac{R}{C_p}\right)}{1 - \frac{\alpha R}{C_p}}} -$$

$$-\frac{\alpha+\phi}{(\alpha+\phi)x-\frac{\alpha R}{c_p}-\frac{\phi\frac{R}{c_p}}{1-\frac{\alpha R}{c_p}}}\left[\frac{1}{2(\alpha+\phi)^2}\left(a+(\alpha+\phi)x\right)^2-\right.$$

$$-\frac{2a+\alpha+1}{(\alpha+\phi)^2}\left(a+(\alpha+\phi)x\right)+\frac{a^2+a(\alpha+1)+\alpha+\phi}{(\alpha+\phi)^2}\,\cdot$$

$$\left.\cdot\ln\left(-a-(\alpha-\phi)x\right)x\right] \qquad (27)$$

where $a=-1+\dfrac{\frac{R}{c_p}\phi}{1-\frac{\alpha R}{c_p}}=-1+\phi\delta$. For any physical situation $\left(a+(\alpha+\phi)x\right)$ appears to be negative.

b. <u>Higher Order Terms</u>

Since the substitution of (27) in (23) leads to a quadrature which offers little advantage over direct numerical integration in the physical plane for a specific geometry device, further attention is directed only to the simple case of constant area which is integrable directly.

a) Constant area $(\psi=0)$ no friction $(\phi=0)$

When $\psi=0$, $u=x$, $p=p_0$ the general phase plane trajectories are shown in Fig. 4a of Ref. 4. With these simplifications

$$P_{(u)} = \frac{P_{in}}{u_{in} - \frac{u_{in}^2}{2}} \; \frac{u_{in} - \frac{R}{C_p}}{u - \frac{R}{C_p}} - \frac{\frac{u^2}{2} - u}{u - \frac{R}{C_p}} \tag{28}$$

length can be introduced again by using (15) and (28).

$$a_1 \frac{du}{dz} = \frac{(au - 1)\left(u - \frac{R}{C_p}\right)}{\dfrac{\dfrac{P_{in}\left(u_{in} - \frac{R}{C_p}\right)}{u_{in} - \frac{u_{in}^2}{2}} - \frac{u^2}{2} + u}{u - \frac{R}{C_p}} - \left(1 - \frac{C_p}{R}\right) u}$$

$$dz = \frac{a_1 \left[\dfrac{P_{in}\left(u_{in} - \frac{R}{C_p}\right)}{u_{in} - \frac{u_{in}^2}{2}} - \left(\frac{3}{2} - \frac{C_p}{R}\right) u^2 + \frac{R}{C_p} u\right] du}{(au - 1)\left(u - \frac{R}{C_p}\right)^2}$$

$$= a_1 \left[\frac{A_1}{\left(u - \frac{R}{C_p}\right)^2} + \frac{A_2}{\left(u - \frac{R}{C_p}\right)} + \frac{B}{(au - 1)}\right] du$$

where

$$A_1 = \frac{-\frac{1}{2}\left(\frac{R}{C_p}\right)^2 + \frac{R}{C_p} + \dfrac{P_{in}\left(u_{in} - \frac{R}{C_p}\right)}{u_{in} - \frac{u_{in}^2}{2}}}{\frac{aR}{C_p} - 1}$$

$$A_2 = \frac{-\frac{1}{2}\left(\frac{R}{C_p}\right)^2 + \frac{3R}{C_p} - 2 + a\,\dfrac{P_{in}\left(u_{in} - \frac{R}{C_p}\right)}{u_{in} - \frac{u_{in}^2}{2}}}{\left(\frac{aR}{C_p} - 1\right)^2}$$

$$B = \frac{-\frac{3}{2} + \frac{C_p}{R} + \frac{aR}{C_p} + a^2\,\dfrac{P_{in}\left(u_{in} - \frac{R}{C_p}\right)}{u_{in} - \frac{u_{in}^2}{2}}}{\left(1 - \frac{aR}{C_p}\right)^2}$$

$$z = a_1\left[C_2 + \frac{A_1}{\left(u - \frac{R}{C_p}\right)} + A_2 Ln\left(u - \frac{R}{C_p}\right) + \frac{B}{a} Ln\,(au-1)\right] \quad (29)$$

C_2 is of course determined by the condition that at the inlet $z = 0$, $P = P_{inlet}$, $u = u_{inlet}$

$C_2 = -\left[\text{The rest of the right hand side}\right]$ evaluated at $u = u_{in}$.

b) $\psi = 0$ (constant area) $\phi \ll 1$ (small friction) the general phase plane for this case is shown in Fig. 4b of Ref. 4. Supersonic acceleration in this case is confined to every small region of the phase plane and can occur only when $\phi < 1/4\gamma(\gamma-1)$.

Then

$$P_{(u)} = \frac{1}{(\alpha+\phi)u - \frac{\phi}{\left(\frac{C_p}{R}-\alpha\right)} - \frac{\alpha R}{C_p}} \Bigg[C_1 - \frac{1}{2(\alpha+\phi)} (a$$

$$+ (\alpha+\phi)\,u)^2 + \frac{2a+\alpha+1}{\alpha+\phi}\,(a + (\alpha+\phi)u)$$

$$- \frac{a^2 + a(\alpha+1) + (a+\phi)}{\alpha+\phi}\, Ln - [a + (\alpha+\phi)u] \Bigg] \qquad (30)$$

where

$$C_1 = P_{in}\left[(\alpha+\phi)\,u_{in} - \frac{\phi}{\frac{C_p}{R}-\alpha} - \frac{\alpha R}{C_p}\right]$$

$$+ \frac{1}{2(\alpha+\phi)}\,(a + (\alpha+\phi)u_{in})^2 - \frac{2a+\alpha+1}{\alpha+\phi}\,(a + (\alpha+\phi)u_{in})$$

$$+ \frac{a^2 + a(\alpha+1) + \alpha + \phi}{\alpha+\phi}\, Ln - [a + (\alpha+\phi)u_{in}]$$

For this case (15) gives

$$a_1 \frac{du}{dZ} = \frac{(\alpha u - 1)\left(u - \frac{R}{C_p}\right) + \phi u^2}{P - \left(\frac{C_p}{R} - 1\right)u}$$

$$dz = a_1 \frac{P - \left(\frac{C_p}{R} - 1\right)u}{(\alpha u - 1)\left(u - \frac{R}{C_p}\right) + \phi u^2}\, du \qquad (31)$$

neglecting powers of ϕ greater than the first (30) becomes:

$$P_{(u)} = \frac{1}{(a+\phi)u - \frac{aR}{C_p} - \phi\delta} \left[C_1 - \frac{(au-1)^2 + 2(au-1)\phi\delta}{2(a+\phi)} \right.$$

$$+ \frac{a-1+2\phi\delta}{a+\phi} (au - 1 + \phi\delta)$$

$$\left. - \frac{\phi}{a+\phi} (a\delta + \delta + 1) \, Ln\,(1 - au) \right] \tag{32}$$

where

$$\delta = \frac{\frac{R}{C_p}}{\left(1 - \frac{aR}{C_p}\right)}$$

integrating (31) with (32) we obtain

$$z = a_1 \left[C_2 - \left(\frac{C_p}{R} - 1\right) I_1 + C_1 I_2 - \frac{1}{2(a+\phi)} I_3 \right.$$

$$\left. + \frac{a-1+2\phi\delta}{a+\phi} I_4 - \frac{\phi(a\delta + \delta + 1)}{a+\phi} I_5 \right] \tag{33}$$

where all the integrals can be found explicitly except I_5 which requires the approximation $Ln\,(1 - au) \approx -(au + \epsilon(au)^2)$ where ϵ is taken to give the best approximation to the Ln over the range of au of interest (see Appendix II). For $au \leq .5$ $\epsilon = 1/2$ gives about 4 per cent error under the worst condition $(au = .5)$. Since this term is of order ϕ the error introduced in Z should be small. The above integrals are as follows:

$$I_1 = \int \frac{du}{\left[(a+\phi)u^2 - \left(1 + \frac{aR}{C_p}\right)u + \frac{R}{C_p}\right]}$$

$$I_2 = (a+\phi)\int\left[\frac{A_{12}}{\left[(a+\phi)u - \left(\frac{aR}{C_p} + \phi\delta\right)\right]^2} + \frac{A_{22}}{\left[(a+\phi)u - \frac{aR}{C_p} + \phi\delta\right]} + \frac{B_2}{\left[(a+\phi)u - (1-\phi\delta)\right]}\right]du$$

$$I_3 = (a+\phi)\int\left[\frac{A_{13}}{[\quad]} + \frac{A_{23}}{[\quad]} + \frac{B_3}{[\quad]}\right]du$$

$$I_4 = (a+\phi)\int\left[\frac{A_{14}}{[\quad]} + \frac{A_{24}}{[\quad]} + \frac{B_4}{[\quad]}\right]du$$

$$I_5 = (a+\phi)\int\left[\frac{A_{15}}{[\quad]} + \frac{A_{25}}{[\quad]} + \frac{B_5}{[\quad]}\right]du$$

where

$$A_{21} = \frac{1}{\frac{\alpha R}{C_p} - 1 + 2\phi\delta}$$

$$A_{22} = \frac{\alpha + \phi}{\left(\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right)^2}$$

$$B_2 = \frac{1}{\left(\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right)^2}$$

$$A_{13} = \frac{\alpha\left[\frac{\alpha R}{C_p}(1-2\alpha) + \alpha + \phi\left\{\delta(1-2\alpha) + 2\frac{\alpha R}{C_p}(\alpha\delta - 1) + 2(1-\alpha\delta)\right\}\right]}{\left[\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right](\alpha+\phi)^2}$$

$$A_{23} = \frac{\alpha^3\left(\frac{R}{C_p}\right)^2 + 3\alpha^2 + \phi\left[4\alpha^2\frac{R}{C_p}\delta - 6\alpha^2\delta - 2\alpha\delta + 4\alpha\right]}{\left[\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right]^2(\alpha+\phi)}$$

$$B_3 = \frac{a(1-a) + \phi\left[2a^2\delta - 2a\delta\right]}{\left[\frac{aR}{C_p} - 1 + 2\phi\delta\right]^2 (a+\phi)^2}$$

$$A_{14} = \frac{\left[a^2\frac{R}{C_p} - a + \phi(2a\delta - 1)\right](a+\phi)}{\left[\frac{aR}{C_p} - 1 + 2\phi\delta\right] (a+\phi)^2}$$

$$A_{24} = \frac{\phi\left[2\frac{aR}{C_p} - 1\right](a+\phi)}{\left[\frac{aR}{C_p} - 1 + 2\phi\delta\right]^2 (a+\phi)}$$

$$B_4 = \frac{\phi a\left[\frac{aR}{C_p} - 1\right]}{\left[\frac{aR}{C_p} - 1 + 2\phi\delta\right]^2 (a+\phi)^2}$$

$$A_{15} = \frac{\epsilon a^4\left(\frac{R}{C_p}\right)^2 + a^3\frac{R}{C_p} + \phi\left[a^2\delta + \frac{a^2R}{C_p} + 2\epsilon\frac{a^3R}{C_p}\delta\right]}{\left[\frac{aR}{C_p} - 1 + 2\phi\delta\right] (a+\phi)^2}$$

$$A_{25} = -\frac{(\alpha+\phi)\left[(\alpha+\epsilon)\left(\frac{\alpha R}{C_p}\right)^2\left(\alpha^2 - 2\frac{\alpha R}{C_p} + 2\right) - \phi\left\{\alpha\delta + \epsilon\left[4\alpha\left(\frac{R}{C_p}\right)^2 - 4\alpha^2\left(\frac{R}{C_p}\right)^3 + 8\delta\left(\frac{\alpha R}{C_p}\right)^2 - 4\delta\frac{\alpha R}{C_p} - 2\delta\frac{\alpha^4 R}{C_p}\right]\right\}\right]}{\left(\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right)^2 (\alpha+\phi)}$$

$$B_5 = \frac{\alpha^2(1-\delta+\epsilon) + \phi(\alpha - 2\alpha^2\epsilon\delta)}{\left(\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right)^2 (\alpha+\phi)^2}$$

with the above substitutions (33) becomes:

$$Z = a_1\left\{C_2 - \frac{\frac{C_p}{R} - 1}{2(\alpha+\phi)}\left[Ln\left[(\alpha+\phi)u^2 - \left(1+\frac{\alpha R}{C_p}\right)u + \frac{R}{C_p}\right]\right.\right.$$

$$\left. + \frac{\left(1+\frac{\alpha R}{C_p}\right)\left(1 + \frac{2\phi\delta}{1-\frac{\alpha R}{C_p}}\right)}{\left(1-\frac{\alpha R}{C_p}\right)} Ln\left[\frac{(\alpha+\phi)u - 1 + \phi\delta}{(\alpha+\phi)u - \frac{\alpha R}{C_p} - \phi\delta}\right]\right] -$$

$$- C_1 \left[\frac{1}{\left(\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right)\left((\alpha+\phi)u - \frac{\alpha R}{C_p} - \phi\delta\right)} \right.$$

$$+ \frac{\alpha+\phi}{\left[\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right]^2} Ln\left[(\alpha+\phi)u - \frac{\alpha R}{C_p} - \phi\delta\right]$$

$$\left. - \frac{1}{\left[\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right]^2} Ln\left[(\alpha+\phi)u - (1-\phi\delta)\right] \right]$$

$$- \frac{\{1\}}{\left[(\alpha+\phi)u - \frac{\alpha R}{C_p} - \phi\delta\right]\left[\frac{\alpha R}{C_p} - 1 + 2\phi\delta\right](\alpha+\phi)^3} +$$

$$+\ \frac{\{2\}\ Ln\left[(\alpha+\phi)u-\frac{\alpha R}{C_p}-\phi\delta\right]}{\left[\frac{\alpha R}{C_p}-1+2\phi\delta\right]^2(\alpha+\phi)^2}$$

$$+\ \frac{\{3\}\ Ln\left[(\alpha+\phi)u-(1-\phi\delta)\right]}{\left[\frac{\alpha R}{C_p}-1+2\phi\delta\right]^2(\alpha+\phi)^3}\Bigg\} \qquad (34)$$

where $C_2 = -\left[\text{everything else on the right hand side}\right]$ evaluated at $u = u_{inlet}$. The $\{\}$'s are constants involving γ, α, ϕ and ϵ the Logarithmic approximation, specifically

$$\{1\} = \frac{\alpha^2}{2}-\alpha^3+\frac{R}{C_p}\left(\alpha^4-\frac{\alpha^2}{2}\right)+\phi\Bigg[\alpha-2\alpha^2(1-\alpha\delta+\delta)$$

$$-\frac{1}{2}\alpha\delta-\alpha^4\delta\,\frac{R}{C_p}-\epsilon\,\alpha^4\left(\frac{R}{C_p}\right)^2(\alpha\delta+\delta+1)\Bigg]$$

$$\{2\} = -\frac{\alpha}{2}\left[\left(\frac{\alpha R}{C_p}\right)^2 + 3\alpha\right] + \phi\left[\epsilon\alpha^3\left(\frac{R}{C_p}\right)^2\left(\alpha^2 - 2\frac{\alpha R}{C_p} + 2\right)(\alpha\delta + \delta + 1)\right.$$

$$\left. + \alpha\delta(4\alpha + \alpha^2 + 1) - \alpha + 2\frac{\alpha^2 R}{C_p}(\alpha - \delta - 1)\right]$$

$$\{3\} = \frac{\alpha(1-\alpha)}{2} + \phi\left[\alpha\delta(\alpha^2 + 1) - \alpha^2\delta(\alpha\delta + \delta + 1)\right.$$

$$\left. + \epsilon\alpha^2(\alpha\delta + \delta + 1)\right]$$

CHAPTER IV

CONCLUSION

The expressions found by the method of Lighthill for $P(u)$ and $u(z)$ for one dimensional compressible flow at constant area should facilitate the interpretation of the experiments with the constant area TWP with small friction. Although this method produced a quadrature for the first order component of velocity due to area change, it seems doubtful that evaluating this expression would result in any simplification over direct numerical solution for a specific case.

The results also indicate the regions of operation in the local Mach number plane. This plane illustrates the conditions that must be satisfied to increase both Mach number and velocity in this kind of process. This curve seems to offer the suggestion that, according to the one-dimensional flow it will be necessary to increase the phase velocity of the wave as it travels down the accelerator. The one-dimensional approximation also indicates that the performance is improved by making the inlet conditions of the gas as uniform as possible.

The calculations in the appendix indicate that in terms of Dahlberg's nondimensional friction, the friction will be small in devices operating at low pressures. The advantage of maintaining a disequilibrium between electron temperature and gas temperature is apparent, since elevated electron temperature gives increased conductivity without the increase in viscosity which accompanies increased gas temperature. Some data is also presented in Appendix I which serves to compare various gases in terms of the friction parameter ϕ .

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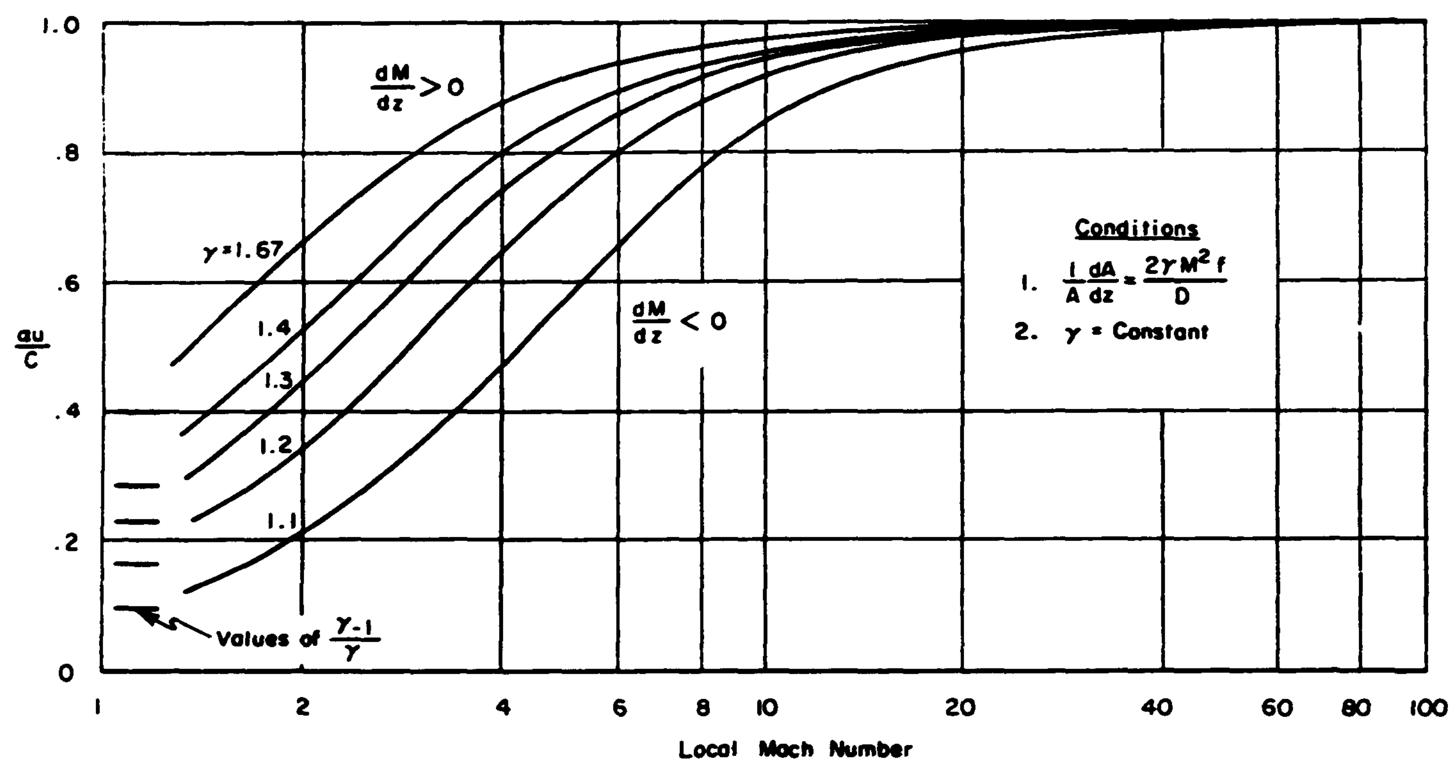

Figure 1. Conditions for local change in Mach number

APPENDIX I

THE NONDIMENSIONAL FRICTION FACTOR

Dahlberg (Ref. 4) defines the friction parameter $\phi = \frac{2\rho u}{B_e{}^2\sigma}\frac{f}{D}$ in terms of the friction factor f. In terms of the skin friction coefficient, C_f , this becomes $\phi = \frac{\rho u C_f}{2 B_e{}^2\sigma}$. C_f can be approximated as the laminar or turbulent skin friction coefficient depending on the magnitude of the Reynolds number, $Re = \frac{\rho u D}{\eta}$. From the perfect gas law it follows that

$$\rho\left(\frac{Kg\,M}{M^3}\right) = \frac{M_\omega\,P_{(mm)}}{T\,(^\circ K)} \times .0166$$

where P is in mm of mercury T is in °K and ρ is in $Kg\,M/M^3$. M_ω is the molecular weight. Using the simplified kinetic theory of Hirschfelder, Curtis and Bird. (Ref. 6)

$$\eta = 2.67 \times 10^{-6}\,\frac{\sqrt{M_\omega T}}{\sigma^2}\,\frac{Kg\,M}{M\,sec}$$

where σ is the equivalent rigid sphere molecular diameter in Angstrom units. Substituting into the Reynolds number we obtain

$$Re = uD\,\frac{P\sqrt{M_\omega}\,\sigma^2}{(T)^{3/2}} \times 6.02 \times 10^3$$

for T at 5000°K and 1 mm pressure with

$$D = .1m \qquad u = 10^4 m/sec \qquad Re = 51$$

For Reynolds numbers of this order the laminar skin friction will certainly be applicable in which case

$$C_f = \frac{1.3}{\sqrt{Re_{diameter}}}$$

With the above relations the friction parameter becomes

$$\phi = \frac{1.34 \times 10^{-4}}{B^2 \sigma_{mho}} \frac{(M_w)^{3/4} \sqrt{u P_{(mm)}}}{\sqrt{D_M}\ \sigma_{A^\circ} (T_{^\circ K})^{1/4}} \quad *$$

for nitrogen at 5000°K and 1 mm pressure with $B = .1$ weber/meter and $\sigma = 100$ mho/m $\phi = 4.06 \times 10^{-3}$

In Table I the appropriate constants for some commonly used gases are given to facilitate a comparison of their performance in the TWP.

TABLE I

Gas	σ_{A°	M_w	Proportional to $\frac{\sqrt{M_w}}{s_A^2}$	$\sqrt{M_w}\,\sigma^2$	$\frac{M_w^{3/4}}{\sigma}$
	Cross Section (Angstroms)	Molecular Weight	(η)	(Re)	(ϕ)
N_2	3.75	28	.376	74.5	3.25
He	2.18	4	.42	9.5	1.3
A	3.64	40	.48	84	4.4
H_2	2.92	2	.166	12.1	.57
Air	3.7	29	.394	74	3.4
Hg	3.56	200	1.14	180	15.1
N	1.68	14	1.33	10.5	4.3
CO_2	3.89	44	.44	100	4.37
CH_4	3.79	16	.278	57.5	2.1

From the above table it appears that hydrogen has by far the best characteristics as far as the frictional losses are concerned. For most of the cases of interest the small ϕ assumption appears to be justified.

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* This equation can also be written in terms of the Mach number, u/a. It becomes

$$\phi = \frac{1.34 \times 10^{-3}}{\sigma B^2 \sqrt{D}} \frac{\sqrt{M_w}\ \sqrt[4]{\gamma}}{\sigma_{A^\circ}} \quad \sqrt{PM}$$

APPENDIX II

APPROXIMATION TO THE LOGARITHMIC FUNCTION

In order to approximate I_S, consider the first two terms in the series for $\ln(1-\alpha u) = -\left[(\alpha u) + \frac{1}{2}(\alpha u)^2 + \frac{1}{3}(\alpha u)^3 \ldots\right]$ In order that the integral be expandable in partial fractions the numerator must be of order 2 or less. We choose the approximation $\ln(1-\alpha u) \approx -[\alpha u + \epsilon(\alpha u)^2]$ where ϵ is matched at a point near the average of αu over the acceleration range of interest. For $\alpha u < .5$ $\epsilon = \frac{1}{2}$ gives a good approximation, the range of validity of a particular value getting worse as $\alpha u = 1$ is approached. Values of ϵ for a perfect fit at a given value of αu are given below.

for $\ln(1-\alpha u) = -\alpha u - \epsilon(\alpha u)^2$

αu	ϵ
.4	.69
.5	.77
.6	.89
.7	1.02
.8	1.25
.9	1.48
.95	2.28

APPENDIX III

THE FIELD-VELOCITY WEIGHTING FUNCTION

Because the fields in the TWP vary so strongly with radius, if KR_o is large the field-velocity weighting function, α , must be included. In the text α is defined by

$$\alpha u = \frac{\int_0^{R_o} I_1^2(Kr)\ V(r)\ r dr}{\frac{R_o^2}{2}\left[I_1^2(KR_o) - I_0(KR_o) I_2(KR_o)\right]}$$

To predict the performance of a TWP device by means of the one-dimensional approximate analysis, a knowledge of the behavior of α for different velocity profiles, and various magnitudes of KR_o is required.

Assuming the density is constant across the cross section, u , the mass average velocity, is $\frac{2}{R_o^2}\int_0^{R_o} r\,V(r)\,dr$

As a simple representation of a general velocity profile assume

$$V(r) = u_o \qquad 0 \le r \le r_c$$

$$= u_o\left(1 - \frac{r - r_c}{R_o - r_c}\right) \qquad r_c \le r \le R_o$$

as represented in Fig. A-1. Then

$$\frac{u}{u_o} = 1 + \frac{r_c}{R_o}\left(1 + \frac{r_c}{R_o}\right) - \frac{2}{3}\,\frac{1 - \left(\frac{r_c}{R_o}\right)^3}{1 - \frac{r_c}{R_o}}$$

and

$$\alpha = \frac{u_o}{u\left(1 - \frac{r_c}{R_o}\right)}\left\{1 - \frac{\left(\frac{r_c}{R_o}\right)^3\left[I_1^2(Kr_c) - I_0(Kr_c) I_2 Kr_c\right]}{I_1^2(KR_o) - I_0(KR_o) I_2(KR_o)}\right.$$

$$- \frac{2\int_{\frac{r_c}{R_0}}^{1} I_1^2(KR_0 r^*) r^{*2} dr^*}{I_1^2(KR_0) - I_0(KR_0) I_2(KR_0)} \Bigg\}$$

The last term in the $\{ \}$ requires a numerical integration or an approximation to I_1. For small argument $(KR_0 << 1)$

$$I_0(KR_0) \approx 1 + \left(\frac{KR_0}{2}\right)^2$$

$$I_1(KR_0) \approx \frac{KR_0}{2}$$

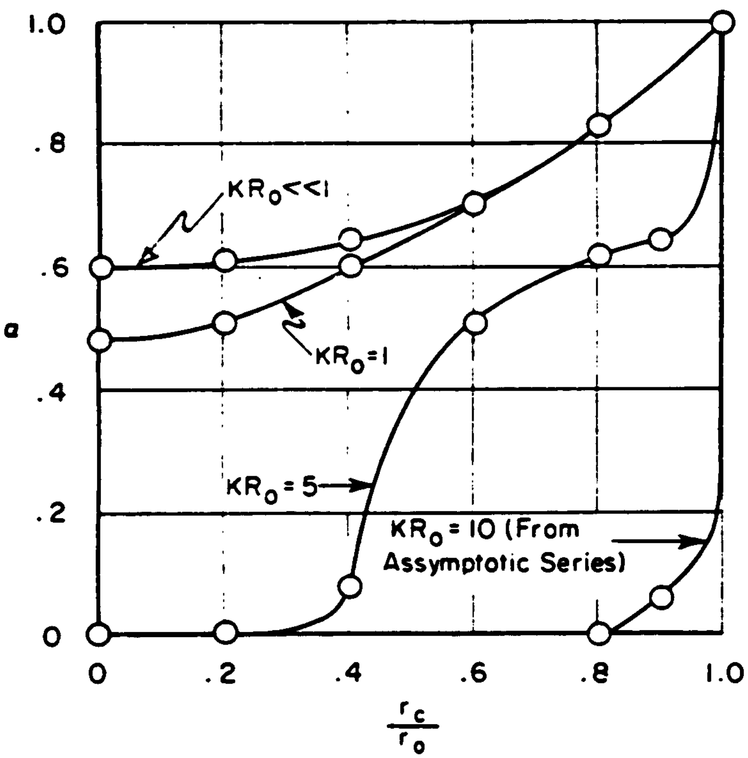
$$I_2(KR_0) \approx \frac{1}{2}\left(\frac{KR_0}{2}\right)^2$$

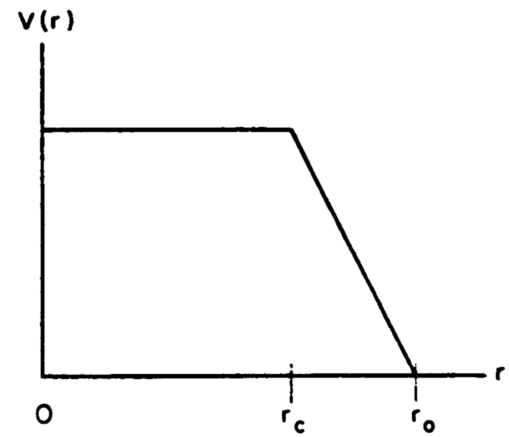
and $$\alpha \approx \frac{u_0}{u\left(1 - \frac{r_c}{R_0}\right)} \left\{ .2\left(1 - \left(\frac{r_c}{R_0}\right)^5\right) \right\}$$

Some values of α *vs.* $\frac{r_c}{R_0}$ for various assumed values of Kr_0 are plotted in Fig. A-1. Note that as $\frac{r_c}{R_0} \rightarrow 1$ α must always approach 1 from below regardless of the value of KR_0, and at $KR_0 > 5$ this approach is very rapid from 0 in the immediate vicinity of $\frac{r_c}{R_0} = 1$ Because of the form of the expression for α the asymptotic approximation to I_1 does not give the right limit as $\frac{r_c}{R_0} \rightarrow 1$.

Figure A-1 indicates that for most applications there would be little advantage in operating above $KR_0 = 2 \rightarrow 4$ unless a very nearly slug velocity profile was present.

$$\alpha = \frac{\int_0^{r_0} I_1^2(r)\, r\, V(r)\, dr}{\int_0^{\delta} \frac{V(r)\, r\, dr}{r_0^2} \cdot \int_0^{r_0} I_1^2(r)\, r\, dr}$$





ASSUMED PROFILE

I_1 = Bessel Function of 1st kind, imaginary argument.

Figure A-1. Illustrations of effect of nonuniform profiles

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One possible technique for accelerating a plasma to a high Mach number is that of passing a magnetic field through the plasma. This technique, called the traveling wave pump, is characterized by a complicated set of differential equations. These equations have been approximated by a one-dimensional steady state form. Several approximate integrals are found for the one-dimensional equations. The results indicate that the viscosity losses

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